Derivation of the Compton Equation

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The 1927 Nobel Prize in Physics was awarded to Arthur Compton for his discovery of the Compton effect. The Compton effect, or Compton scattering, occurs when the incident x-ray photon is deflected from its original path by an interaction with an electron. The electron is ejected from its orbit and the photon loses energy. The change in wavelength of the photon is described by the Compton Equation. The derivation of this equation is presented below:

Consider the situation depicted in Figure 1. A photon is traveling in the positive **i** direction. It then collides with a stationary electron. After the elastic collision, the photon deflects at an angle of θ and the electron deflects at an angle of ϕ.

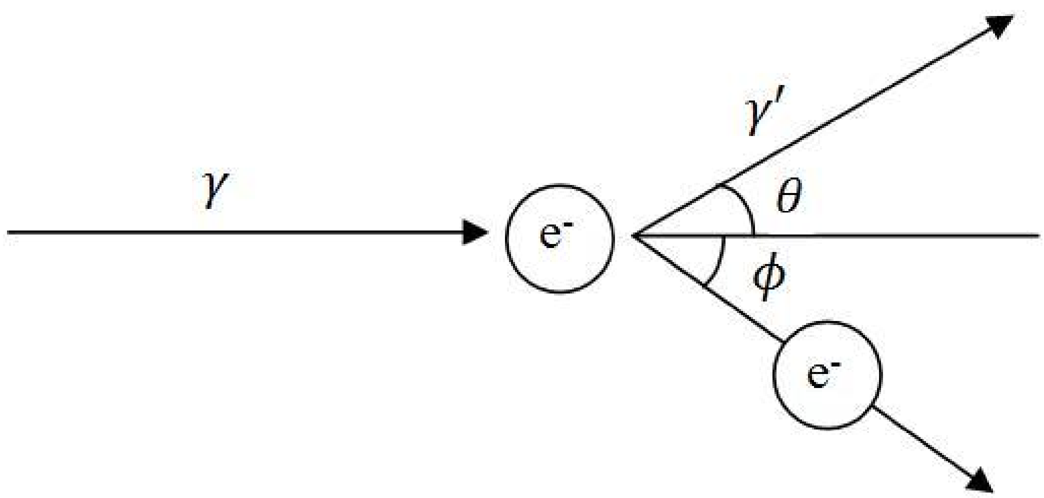

Figure 1: Diagram of the Compton effect.

The following variables will be used:

M_e	mass of an electron
P_γ	momentum of the initial photon
$P_{\gamma'}$	momentum of the final photon
P_e	momentum of the electron
θ	angle of the final photon
ϕ	angle of the electron

During a collision, energy is always conserved:

$$E_\gamma + E_e = E_{\gamma'} + E_{e'}.$$

Next we expand and simplify:

$$P_\gamma c + M_e c^2 = P_{\gamma'} c + \sqrt{P_e^2 c^2 + M_e^2 c^4}$$
$$P_\gamma c + M_e c^2 = P_{\gamma'} c + c\sqrt{P_e^2 + M_e^2 c^2}$$
$$P_\gamma + M_e c = P_{\gamma'} + \sqrt{P_e^2 + M_e^2 c^2}. \tag{1}$$

Conserving the **i** components of momentum yields

$$P_\gamma = P_e \cos(\phi) + P_{\gamma'} \cos(\theta).$$

Rearranging:

$$P_\gamma - P_{\gamma'} \cos(\theta) = P_e \cos(\phi). \tag{2}$$

Conserving the **j** components of momentum yields

$$0 = P_e \sin(\phi) - P_{\gamma'} \sin(\theta).$$

Rearranging:

$$P_{\gamma'} \sin(\theta) = P_e \sin(\phi). \tag{3}$$

Squaring (2) and (3) and then adding yields:

$$P_\gamma^2 - 2P_\gamma P_{\gamma'} \cos(\theta) + P_{\gamma'}^2 \cos^2(\theta) + P_{\gamma'}^2 \sin^2(\theta) = P_e^2 \cos^2(\phi) + P_e^2 \sin^2(\phi).$$

Simplifying:

$$P_\gamma^2 - 2P_\gamma P_{\gamma'} \cos(\theta) + P_{\gamma'}^2 = P_e^2. \tag{4}$$

Rearranging terms in (1) yields

$$P_\gamma + M_e c - P_{\gamma'} = \sqrt{P_e^2 + M_e^2 c^2}.$$

Squaring both sides:

$$(P_\gamma + M_e c - P_{\gamma'})^2 = P_e^2 + M_e^2 c^2.$$

Simplifying:

$$P_\gamma^2 + M_e^2 c^2 + P_{\gamma'}^2 + 2P_\gamma M_e c - 2P_\gamma P_{\gamma'} - 2P_{\gamma'} M_e c = P_e^2 + M_e^2 c^2$$
$$P_\gamma^2 + P_{\gamma'}^2 + 2P_\gamma M_e c - 2P_\gamma P_{\gamma'} - 2P_{\gamma'} M_e c = P_e^2. \tag{5}$$

Combining (4) and (5) results in

$$P_\gamma^2 - 2P_\gamma P_{\gamma'} \cos(\theta) + P_{\gamma'}^2 = P_e^2 = P_\gamma^2 + P_{\gamma'}^2 + 2P_\gamma M_e c - 2P_\gamma P_{\gamma'} - 2P_{\gamma'} M_e c.$$

Simplifying:

$$-2P_\gamma P_{\gamma'} \cos(\theta) = 2P_\gamma M_e c - 2P_\gamma P_{\gamma'} - 2P_{\gamma'} M_e c$$
$$2P_\gamma P_{\gamma'} - 2P_\gamma P_{\gamma'} \cos(\theta) = 2P_\gamma M_e c - 2P_{\gamma'} M_e c$$
$$2P_\gamma P_{\gamma'} (1 - \cos(\theta)) = 2M_e c (P_\gamma - P_{\gamma'})$$
$$\frac{P_\gamma - P_{\gamma'}}{P_\gamma P_{\gamma'}} = \frac{1 - \cos(\theta)}{M_e c}$$
$$\frac{1}{P_{\gamma'}} - \frac{1}{P_\gamma} = \frac{1 - \cos(\theta)}{M_e c}. \tag{6}$$

Multiplying (6) by h (Planck's constant $= 6.626 \times 10^{-34}$):

$$\frac{h}{P_{\gamma'}} - \frac{h}{P_\gamma} = \frac{h}{M_e c}(1 - \cos(\theta)). \tag{7}$$

Since $P = \frac{h}{\lambda}$, $\frac{h}{P} = \lambda$. Applying this to (7) yields

$$\Delta\lambda = \lambda_{\gamma'} - \lambda_\gamma = \lambda_C (1 - \cos(\theta)) \tag{8}$$

where $\lambda_C \equiv \frac{h}{M_e c} = 2.43 \times 10^{-12}$ m $= 0.0243$ Å.
Equation (8) is known as the *Compton Equation*.