Derivation of the gradient, divergence, curl, and the Laplacian in Spherical Coordinates

Rustem Bilyalov

November 5, 2010

The required transformation is $x, y, z \to r, \theta, \phi$. In Spherical Coordinates

$$u^1 = r, \qquad u^2 = \theta, \qquad u^3 = \phi.$$

Also

$$x = x^1 = r\sin(\theta)\cos(\phi) \qquad y = x^2 = r\sin(\theta)\sin(\phi) \qquad z = x^3 = r\cos(\theta).$$

The scale factors are determined as follows:

$$\begin{aligned}
\mathrm{g}_{11} &= \sum_{k=1}^{3}\left(\frac{\partial x^k}{\partial u^1}\right)^2 \\
&= \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 \\
&= \left[\frac{\partial}{\partial r}(r\sin(\theta)\cos(\phi))\right]^2 + \left[\frac{\partial}{\partial r}(r\sin(\theta)\sin(\phi))\right]^2 + \left[\frac{\partial}{\partial r}(r\cos(\theta))\right]^2 \\
&= \sin^2(\theta)\cos^2(\phi) + \sin^2(\theta)\sin^2(\phi) + \cos^2(\theta) \\
&= \sin^2(\theta)(\cos^{(}\phi) + \sin^2(\phi)) + \cos^2(\theta) \\
&= \sin^2(\theta)(1) + \cos^2(\theta) \\
&= 1 = h_1^2
\end{aligned}$$

$$\therefore h_1 = 1.$$

$$\begin{aligned}
\mathrm{g}_{22} &= \sum_{k=1}^{3}\left(\frac{\partial x^k}{\partial u^2}\right)^2 \\
&= \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 \\
&= \left[\frac{\partial}{\partial \theta}(r\sin(\theta)\cos(\phi))\right]^2 + \left[\frac{\partial}{\partial \theta}(r\sin(\theta)\sin(\phi))\right]^2 + \left[\frac{\partial}{\partial \theta}(r\cos(\theta))\right]^2 \\
&= r^2\cos^2(\theta)\cos^2(\phi) + r^2\cos^2(\theta)\sin^2(\phi) + r^2\sin^2(\theta) \\
&= r^2(\cos^2(\theta)(\cos^2(\phi) + \sin^2(\phi)) + \sin^2(\theta)) \\
&= r^2(\cos^2(\theta)(1) + \sin^2(\theta)) \\
&= r^2(1) \\
&= r^2 = h_2^2
\end{aligned}$$

$$\therefore h_2 = r.$$

$$
\begin{aligned}
g_{33} &= \sum_{k=1}^{3}\left(\frac{\partial x^k}{\partial u^3}\right)^2 \\
&= \left(\frac{\partial x^1}{\partial u^3}\right)^2 + \left(\frac{\partial x^2}{\partial u^3}\right)^2 + \left(\frac{\partial x^3}{\partial u^3}\right)^2 \\
&= \left[\frac{\partial}{\partial \phi}(r\sin(\theta)\cos(\phi))\right]^2 + \left[\frac{\partial}{\partial \phi}(r\sin(\theta)\sin(\phi))\right]^2 + \left[\frac{\partial}{\partial \phi}(r\cos(\theta))\right]^2 \\
&= r^2\sin^2(\theta)\sin^2(\phi) + r^2\sin^2(\theta)\cos^2(\phi) + 0 \\
&= r^2\sin^2(\theta)(\sin^2(\phi)+\cos^2(\phi)) \\
&= r^2\sin^2(\theta)(1) \\
&= r^2\sin^2(\theta) = h_3^2
\end{aligned}
$$

$$\therefore h_3 = r\sin(\theta).$$

The gradient in any coordinate system can be expressed as

$$\nabla = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u^1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u^2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u^3}.$$

The gradient in Spherical Coordinates is then

$$\nabla = \frac{\partial}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

The divergence in any coordinate system can be expressed as

$$\nabla \cdot V = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u^1}(h_2h_3V_1) + \frac{\partial}{\partial u^2}(h_1h_3V_2) + \frac{\partial}{\partial u^3}(h_1h_2V_3)\right]$$

The divergence in Spherical Coordinates is then

$$
\begin{aligned}
\nabla \cdot V &= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial r}(r^2\sin(\theta)V_r) + \frac{\partial}{\partial \theta}(r\sin(\theta)V_\theta) + \frac{\partial}{\partial \phi}(rV_\phi)\right] \\
&= \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial r}(r^2\sin(\theta)V_r) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}(r\sin(\theta)V_\theta) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \phi}(rV_\phi) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}(\sin(\theta)V_\theta) + \frac{1}{r\sin(\theta)}\frac{\partial V_\phi}{\partial \phi}
\end{aligned}
$$

The curl in any coordinate system can be expressed as

$$
\begin{aligned}
\nabla \times \boldsymbol{V} &= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\boldsymbol{e}}_1 & h_2\hat{\boldsymbol{e}}_2 & h_3\hat{\boldsymbol{e}}_3 \\ \frac{\partial}{\partial u^1} & \frac{\partial}{\partial u^2} & \frac{\partial}{\partial u^3} \\ h_1V_1 & h_2V_2 & h_3V_3 \end{vmatrix} \\
&= \frac{1}{h_1h_2h_3}\left(h_1\hat{\boldsymbol{e}}_1\left(\frac{\partial}{\partial u^2}h_3V_3 - \frac{\partial}{\partial u^3}h_2V_2\right) - h_2\hat{\boldsymbol{e}}_2\left(\frac{\partial}{\partial u^1}h_3V_3 - \frac{\partial}{\partial u^3}h_1V_1\right) + h_3\hat{\boldsymbol{e}}_3\left(\frac{\partial}{\partial u^1}h_2V_2 - \frac{\partial}{\partial u^2}h_1V_1\right)\right) \\
&= \frac{1}{h_2h_3}\hat{\boldsymbol{e}}_1\left(\frac{\partial}{\partial u^2}h_3V_3 - \frac{\partial}{\partial u^3}h_2V_2\right) - \frac{1}{h_1h_3}\hat{\boldsymbol{e}}_2\left(\frac{\partial}{\partial u^1}h_3V_3 - \frac{\partial}{\partial u^3}h_1V_1\right) + \frac{1}{h_1h_2}\hat{\boldsymbol{e}}_3\left(\frac{\partial}{\partial u^1}h_2V_2 - \frac{\partial}{\partial u^2}h_1V_1\right)
\end{aligned}
$$

The curl in Spherical Coordinates is then

$$
\begin{aligned}
\nabla \times \boldsymbol{V} &= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial \theta}(r\sin(\theta)V_\phi) - \frac{\partial}{\partial \phi}(rV_\theta)\right]\hat{\boldsymbol{r}} - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}(r\sin(\theta)V_\phi) - \frac{\partial V_r}{\partial \phi}\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}} \\
&= \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial \theta}(\sin(\theta)V_\phi) - \frac{\partial V_\theta}{\partial \phi}\right]\hat{\boldsymbol{r}} + \frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial V_r}{\partial \phi} - \frac{\partial}{\partial r}(rV_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}
\end{aligned}
$$

The Laplacian in any coordinate system can be expressed as

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u^1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial u^2}\right)\frac{\partial}{\partial u^3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u^3}\right)\right]$$

The Laplacian in Spherical Coordinates is then

$$
\begin{aligned}
\nabla^2 &= \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial r}\left(r^2\sin(\theta)\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \phi}\left(\frac{1}{\sin(\theta)}\frac{\partial}{\partial \phi}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2} \\
&= \frac{1}{r^2}\left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\left(\cos(\theta)\frac{\partial}{\partial \theta} + \sin(\theta)\frac{\partial^2}{\partial \theta^2}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r^2\sin(\theta)}\frac{\partial}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2}
\end{aligned}
$$